Modular functors from non-semisimple 3d TFTs

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A prominent origin of topological modular functors comes from certain 3d topological field theories (TFTs). This relation can be seen as the starting point of the CFT/TFT correspondence.

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Can we go beyond the semisimple setting?

Main result

Theorem [H., Runkel]
Let $\mathcal{C}$ be a not necessarily semisimple modular tensor category, then the 3d TFT $\mathcal{Z}_{\mathcal{C}}$ of [DGGPR1] constructed from $\mathcal{C}$ induces a topological modular functor:

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Moreover this modular functor coincides with the one constructed from the same input data via generators and relations in [Lyu].

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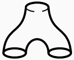
 $\longmapsto \mathrm{Rep}(\mathrm{V})$ (possible field insertions)

 $\longmapsto \mathrm{Hom}_{\mathrm{V}}(M \otimes N, L)$ (3-point blocks on sphere)

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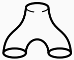
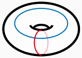
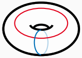
 $\longmapsto \mathrm{Hom}_{\mathrm{V}}(M \otimes N, L)$ (3-point blocks on sphere)

 $\longmapsto \mathrm{Hom}_{\mathrm{V}}(\mathbb{1}, \mathcal{L})$

$\Downarrow \longmapsto \Downarrow \mathcal{S}$ (modular S-transformation)

 $\longmapsto \mathrm{Hom}_{\mathrm{V}}(\mathbb{1}, \mathcal{L})$

Main result II

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Proposition
Let Σ be a (possibly connected) surface with at least one incoming and outgoing boundary component, and let Σ_{gl} be the surface obtained from gluing these boundaries. Then there is a natural isomorphism

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induced by a 3-dimensional bordism.

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The symbol $\oint^{X\in\mathcal{C}}$ is a categorical notion and can be thought of as a precise way to sum over intermediate states.

References

[DGGPR1] Marco De Renzi, Azat M. Gainutdinov, Nathan Geer, Bertrand Patureau-Mirand, and Ingo Runkel. “3-dimensional TQFTs from non-semisimple modular categories”. In: *Selecta Mathematica* 28.2 (2022), 42. arXiv: `1912.02063 [math.GT]`.

[DGGPR2] Marco De Renzi, Azat M. Gainutdinov, Nathan Geer, Bertrand Patureau-Mirand, and Ingo Runkel. “Mapping class group representations from non-semisimple TQFTs”. In: *Commun. Contemp. Math.* 25.01 (2023), 2150091. arXiv: `2010.14852 [math.GT]`.

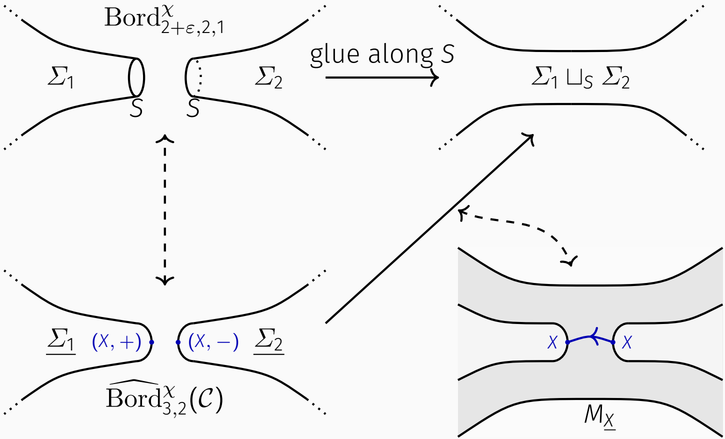
[FRS] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. “TFT construction of RCFT correlators I: partition functions”. In: *Nuclear Physics B* 646.3 (2002), 353–497. arXiv: `0204148 [hep-th]`.

[Lyu] Volodymyr Lyubashenko. “Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity”. In: *Communications in Mathematical Physics* 172 (09 1995), 467–516.

Questions?

Proof sketch I

The relevant 3-bordism can be obtained as follows:

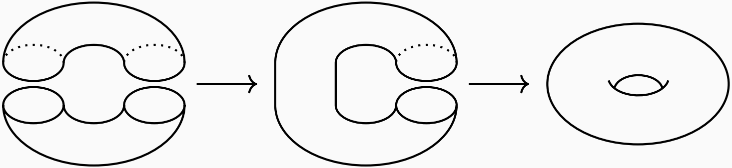


Proof sketch II

We need to distinguish the following two scenarios:

1. We glue boundary components on two different components of Σ.
2. We glue boundary components on a connected component of Σ.

Both of these scenarios are different from a global perspective, as can be seen by the example of gluing two disjoint cylinders to a torus:



Why are non-semisimple theories interesting?

From a physics perspective:

- Applications in statistical physics, e.g. critical dense polymers.
- Wess-Zumino-Witten models with supergroup target are often non-semisimple.
- Twists of supersymmetric QFTs are usually non-semisimple, even derived.

From a mathematics perspective:

- Many 2d TFTs are non-semisimple.
- Can we understand other non-semisimple CFT constructions from the 3d perspective?
- Stronger topological invariants.
- Topological interpretation of algebraic structures.
- Step towards derived TFTs.